ALGEBRAIC CURVES
EXERCISE SHEET 4

Unless otherwise specified, k is an algebraically closed field.

Exercise 4.1. Show that all local rings of the affine line $\mathbb{A}^1_k$ are isomorphic to the same ring R.

Exercise 4.2. An *affine algebraic group* is an affine variety G, whose underlying set is a group, such that the morphisms $i : G \to G,\ g \mapsto g^{-1}$ and $m : G \times G \to G,\ (g,h) \mapsto gh$ are polynomial maps. Let $V_1 = \mathbb{A}^1_k - \{0\}$ and $V_2 = V(xy-1)$. From the first exercise, we call R the local ring of $\mathbb{A}^1_k$ at any point.

(1) Show that $\mathcal{O}(V_1) = k[x, x^{-1}] = k[x,y]/(xy-1)$.
(2) Construct a morphism $V_2 \to \mathbb{A}^1_k$ whose image is V_1.
(3) Show that the local ring of V_2 at any point is isomorphic to R. Are V_2 and $\mathbb{A}^1_k$ isomorphic?
(4) Show that V_2 can be endowed with a structure of affine algebraic group.

Exercise 4.3. Let $V = V(y^2 - x^3)$. Let $\varphi : \mathbb{A}^1_k \to V$ be the morphism defined by $\varphi(t) = (t^2, t^3)$. From the first exercise, we call R the local ring of $\mathbb{A}^1_k$ at any point.

(1) Show φ is a bijective morphism, but is not an isomorphism.
(2) Let $P \in V$. Is the local ring of V at P isomorphic to R?

Exercise 4.4. Let $V = V(Y^2 - X^2(X+1))$ and x, y the residues of X, Y in $\Gamma(V)$. Let $z = \frac{y}{x} \in k(V)$. Find the poles of z and z^2.

Exercise 4.5. Let V be an affine variety and $f \in k(V)$ a rational function. Show that f defines a continous function $U \to k$, for some non empty open subset $U \subset V$. Furthermore f is uniquely determined by this function.

Exercise 4.6. * Let $F \in k[x,y]$ be an irreducible polynomial of degree at most 2. Show that $V(F)$ is either isomorphic to $V_1 = \mathbb{A}^1_k$ or $V_2 = V(xy-1)$. Specify in which case it is isomorphic to V_1 (resp. V_2). (Hint: Use linear changes of coordinates to eliminate monomials in F)